ISL

ENGINEERING COLLEGE

Approved By AICTE, Affiliated to Osmania University

611821

Unit-IV

Function of Complex variables: Limitand continuity of a function, Differentiability and analyticity, Analytic function Necessary and sufficient conditions for a function to be analytic, Cauchy's Riemann equation in polar form, Hormonic functions, complex integration, Cauchy's integral theorem, extension of Cauchy's integral theorem, Cauchy's integral formula, Cauchy's formula for derivatives. (Without proof)

Unit-V

Residue Calculus: Taylor series, Laurent series, (without proofs) zeros and singularities, residues, residue theorem (without proof) evolution of real integral of the type

((a) $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ (b) $\int_{-\infty}^{\infty} f(x)dx$

Text books:

1. Erwin Kreyszig, *Advanced Engineering Mathematics*, John Wiley, 9th Edition,2012.
2. Dr.B.S. Grewal, *Higher Engineering Mathematics*, Khanna Publications, 43rd Edition,2014.

Reference books

1. R.K. Jain & S.R.K. Iyengar, *Advanced Engineering Mathematics*, Narosa Publications, 4th Edition,2014
2. B.V. Ramana, *Higher Engineering Mathematics*, 23rd reprint, 2015.
3. N. Bali, M. Goyal, A text book of Engineering *Mathematics*, Laxmi publications,2010
   H.K. Dass, Er. Rajnish Varma, *Higher Engineering Mathematics*, Schand Technical Third Edition.

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Head
Department of Mathematics
Osmania University
Hyderabad-500 007

29/6/24

PRINCIPAL
ISL ENGG. COLLEGE
Bandlaguda, Hyd.

Chandrayangutta, Bandlaguda, Hyderabad - 500005, Telangana State.
Email: info@islec.edu.in | Tel: 040-29880083 / 29880084

611830

ISL ENGINEERING COLLEGE

Approved By AICTE, Affiliated to Osmania University

Course Code	Course Title						Core / Elective
24BS103MT ISL24UBS201 MT	**ORDINARY DIFFRENTIAL EQUATIONS AND COMPLEX VARIABLE** (Common to All Branches)(**TENTATIVE**)						Core
Prerequisite	Contact Hours per Week				CIE	SEE	Credits
	L	T	D	P			
-	3	1	-	-	40	60	4

Course Objectives

- To provide an overview of ordinary differential equations
- To provide an overview of higher order differential equations
- To understand the sequence and series
- To study complex variable functions
- To learn how to evaluate improper integrals

Course Outcomes

The students will able to

- solve system of linear equations and eigenvalue problems
- solve certain first order differential equation.
- Solve higher order differential equations
- solve basic problems of complex analysis.
- apply the knowledge to solve improper integrals.

Unit-I

Differential Equations of First Order: Exact Differential equations, Linear differential equations, Reducible to Linear differential equations, Orthogonal trajectories of a given family of curves.

Unit-II

Differential Equations of Higher Orders: Solutions of second and higher order linear homogeneousand non-homogeneous equations with constants coefficients, Method of variation of parameters, Cauchy's Euler equations.

Unit-III

Sequence and series: Sequence and Series, General properties of series, series of positive terms, limit comparison test, D'Alembert ratio test, Cauchy's nth root test, Raabe's test, Integral test, Alternating series, Absolute convergence and Conditional convergence.

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ISL ENGG. COLLEGE
Bandlaguda, Hyd.

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29/6/24

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Head
Department of Mathematics
Osmania University
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TO understand the sequence and series
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Chandrayangutta, Bandiaguda, Hyderabad SOOOOS, Telangana State.
Email: info@lslec.edu.in I Tel: 040-29880083 / 29880084

INTRO

21 February 2025 09:53

$1 \overset{2}{\frown} 3 \overset{2}{\frown} 5 \;\ldots\ldots\; n^{th} = (2n-1) = a + (n-1)d$

$= 1 + (n-1)2 = 2n-2+1 = 2n-1$

$a = 1$

$d = 2$

n^{th}, A.P $= a + (n-1)d$

$a \downarrow$ 1st Num, $d \downarrow$ comm. diff

$\boxed{2} \overset{2}{\frown} 4 \overset{2}{\frown} 6 \;\ldots\ldots\; n^{th} = 2n$

$a = 2$

$a + (n-1)d$

$= 2 + (n-1)2 = 2n-2+2 = 2n$ //

$1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \ldots\ldots\; n^{th} = (2n-1)(2n$

1 3 5 ---- $2n-1$

2 4 6 ... $2n$

$\boxed{1 \cdot 2 \cdot 3} + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 \ldots\ldots\; n^{th} = n(n+1)(n+2)$ //

$1 \overset{1}{\frown} 2 \overset{1}{\frown} 3 \overset{1}{\frown} 4 \overset{1}{\frown} 5 \;\ldots\ldots\; n^{th} = a + (n-1)d$

$= n - 1 + 1 = n$ //

$a = 1 \quad d = 1$

2 3 4 ...... $n^{th} = n+1$

3 4 5 .... $n^{th} = n+2$

$\frac{1 \cdot 2 \cdot 3}{4 \cdot 5} + \frac{2 \cdot 3 \cdot 4}{5 \cdot 6} + \frac{3 \cdot 4 \cdot 5}{6 \cdot 7} + \ldots\ldots\; n^{th} = ?$

$4 \overset{1}{\frown} 5 \overset{1}{\frown} 6 \;\ldots\ldots\; n^{th} = 4 + (n-1)1 = n - 1 + 4 = n+3$

$5 \quad 6 \quad 7 \;\ldots\ldots\; n^{th} = 5 + (n-1)1 = n - 1 + 5 = n+4$

$$n^{th} = \frac{n(n+1)(n+2)}{(n+3)(n+4)} //$$

$\frac{1 \cdot 2 \cdot 3}{4 \cdot 5} \cdot x + \frac{2 \cdot 3 \cdot 4}{5 \cdot 6} x^2 + \frac{3 \cdot 4 \cdot 5}{6 \cdot 7} x^3 + \frac{4 \cdot 5 \cdot 6}{7 \cdot 8} x^4 + \dots n^{th}$

$$n^{th} = \frac{n(n+1)(n+2)}{(n+3)(n+4)} \cdot x^n //$$

$\int (x^2 + 5x + 3y)\, dx = ?$ $\frac{x^3}{3} + 5\frac{x^2}{2} + 3xy$

$e^{\int \frac{1}{x} dx} =$

Unit 01: differential equations of first order

22 February 2025 09:55

Exact D.E

Steps: 1) Compare the given eqⁿ with $M\,dx + N\,dy = 0$ (Std form)

and find M, N, then find $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$

2) the given eqⁿ is Exact, if $\left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right]$

3) $\underset{(y\text{-const})}{\int} M\,dx + \underset{(x\text{-eliminated})}{\int} N\,dy = C$

Q Solve, $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

Sol S-① Compare with $M\,dx + N\,dy = 0$,

$M = (e^y + 1)\cos x$ $\qquad$ $N = e^y \sin x$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[(e^y + 1)\cos x\right]$

$= \frac{\partial}{\partial y}\left[e^y \cos x + \cos x\right]$

$\frac{\partial M}{\partial y} = e^y \cos x + 0$

$\frac{\partial M}{\partial y} = e^y \cos x$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[e^y \sin x\right]$

$\frac{\partial N}{\partial x} = e^y \cos x$

S-2 ∴ Given D.E is Exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

S-3: Genʳˡ solⁿ: $\underset{(y\text{-const})}{\int} M\,dx + \underset{(x\text{-eliminated})}{\int} N\,dy = c$

$N = e^y \sin x$

(y-const) (x-eliminated)

$N = e^y \sin x$

$$\Rightarrow \int_{y\text{-const}} (e^y+1)\cos x \, dx + \int 0 \, dy = c$$

$$= \int_{y\text{-const}} e^y \cos x \, dx + \int \cos x \, dx = c$$

$\int \cos x \, dx = \sin x$

$$= e^y \int \cos x \, dx + \int \cos x \, dx = c$$

$$= e^y [\sin x] + \sin x = c$$

$$= \sin x [e^y + 1] = c$$ is the req^rd sol^n

Non exact : method 01

27 February 2025 14:07

Method 1 :-

Inspection method :-

An integrating factor (I.F) of given eqn $Mdx + Ndy = 0$ can be found by inspection as explained below.

1) $d(xy) = x\,dy + y\,dx$

2) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

3) $d[\log xy] = \frac{x\,dy + y\,dx}{xy}$

4) $d\left[\frac{y}{x}\right] = \frac{x\,dy - y\,dx}{x^2}$

5) $d\left[\frac{y^2}{x}\right] = \frac{2xy\,dy - y^2\,dx}{x^2}$

6) $d\left[\frac{x^2}{y}\right] = \frac{2xy\,dx - y^2\,dy}{y^2}$

7) $d\left[\frac{y^2}{x^2}\right] = \frac{2x^2y\,dy - 2xy^2\,dx}{x^4}$

8) $d\left[\frac{x^2}{y^2}\right] = \frac{2y^2x\,dx - 2yx^2\,dy}{y^4}$

9) $d\left[\tan^{-1}\frac{y}{x}\right] = \frac{x\,dy - y\,dx}{x^2+y^2}$

10) $d\left[\tan^{-1}\frac{x}{y}\right] = \frac{y\,dx - x\,dy}{x^2+y^2}$

11) $d\left[\frac{e^x}{y}\right] = \frac{ye^x\,dx - e^x\,dy}{y^2}$

12) $d\left[\frac{e^y}{x}\right] = \frac{xe^y\,dy - e^y\,dx}{x^2}$

13) $d\left[\log(x/y)\right] = \frac{y\,dx - x\,dy}{xy}$.

14) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$

Solve $x\,dy - y\,dx = xy^2\,dx$

Dividing the eq with 'y^2' on B.S,,

Dividing the eq with y^2 on B.S,,

$$\frac{x\,dy - y\,dx}{y^2} = \frac{x\cancel{y^2}}{\cancel{y^2}}\,dx$$

$$\frac{x\,dy - y\,dx}{y^2} = x\,dx$$

$$\Rightarrow \quad x\,dx - \left[\frac{x\,dy - y\,dx}{y^2}\right] = 0$$

$$\Rightarrow \quad x\,dx - \frac{x\,dy + y\,dx}{y^2} = 0$$

{∵ from formula} ∵ 2

$$\Rightarrow \quad x\,dx + \frac{y\,dx - x\,dy}{y^2} = 0$$

$$\Rightarrow x\,dx + d\left[\frac{x}{y}\right] = 0 \qquad \left\{\because d\left[\frac{x}{y}\right] = \frac{y\,dx - x\,dy}{y^2}\right\}$$

⇒ Integrate on B.S,,

$$\int x\,dx + \int d\left[\frac{x}{y}\right] = 0$$

$$\frac{x^2}{2} + \frac{x}{y} = C$$

∵ C is the int const,,

Non exact: 02 I.F

27 February 2025 14:08

$$\text{Non-Exact D.E} \longrightarrow \text{Exact D.E}$$

Non-Exact D.E → Integrating factor (I.F) → Exact D.E

Steps or working Rule!

1) compare with std form $M\,dx + N\,dy = 0$, if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then D.E is Non-Exact, find I.F

2) $I.F = \frac{1}{Mx + Ny}$; $Mx + Ny \neq 0$

Now, $I.F \times (M\,dx + N\,dy) = 0$

then, $M'\,dx + N'\,dy = 0$

3) follow the same steps from S-1, 2, if Exact $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$ then find Gen^rl Sol^n

4) Gen^rl Sol^n! $\underbrace{\int M'\,dx}_{(y\text{-const})} + \underbrace{\int N'\,dy}_{(x\text{-eliminated})} = C$

#Q. Solve, $x^2y\,dx - (x^3+y^3)\,dy = 0$

Sol S-1: compare $M\,dx + N\,dy = 0$

$M = x^2y$ $\quad\quad$ $N = -(x^3+y^3)$

$\frac{\partial M}{\partial y} = x^2$ $\quad\quad$ $\frac{\partial N}{\partial x} = -3x^2$

S-2 : Given D.Eq^n is Non-Exact → I.F, ↓ Exact

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{(x^2y)x + (-(x^3+y^3))y} = \frac{1}{\cancel{x^3y} - \cancel{x^3y} - y^4}$$

$$I.F = -\frac{1}{y^4}$$

Now, $I.F \times (M\,dx + N\,dy) = 0$

$$\Rightarrow -\frac{1}{y^4} \times \left[x^2y\,dx - (x^3+y^3)\,dy \right] = 0$$

$$\Rightarrow \frac{-x^2y}{y^4}\,dx + \frac{(x^3+y^3)}{y^4}\,dy = 0$$

$$\Rightarrow \frac{-x^2}{y^3}\,dx + \left[\frac{x^3}{y^4} + \frac{1}{y} \right] dy = 0$$

Now, Compare with $M'dx + N'dy = 0$

$$M' = -\frac{x^2}{y^3} \qquad N' = \frac{x^3}{y^4} + \frac{1}{y}$$

$$\frac{\partial M'}{\partial y} = -x^2 \frac{\partial}{\partial y}\left[\frac{1}{y^3}\right]$$

$$= -x^2 \frac{\partial}{\partial y}\left[y^{-3}\right]$$

$$= -x^2 \left[-3 \cdot y^{-3-1}\right]$$

$$= -x^2 \times (-3) \cdot y^{-4}$$

$$\frac{\partial M'}{\partial y} = \frac{3x^2}{y^4}$$

$$\frac{\partial N'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^3}{y^4} + \frac{1}{y}\right]$$

$$= \frac{1}{y^4}\frac{\partial}{\partial x}(x^3) + 0$$

$$= \frac{1}{y^4}(3x^2)$$

$$\therefore \frac{\partial N'}{\partial x} = \frac{3x^2}{y^4}$$

S-3: obtained D-Eq^n is Exact, $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$, Now, find Gen^{rl} sol^n,

$$\int M'\,dx + \int N'\,dy = c$$

$$\int M' \, dx + \int N' \, dy = c$$

(y-const) (x-eliminated)

$N' = \left[\frac{x^3}{y^{\cancel{4}}} + \frac{1}{y}\right]$

$$= \int -\frac{x^2}{y^3} \, dx + \int \frac{1}{y} \, dy = c$$

$\left[\because \int \frac{1}{y} dy = \log y\right.$

$\int x^2 dx = \frac{x^3}{3}$

$$= -\frac{1}{y^3} \int x^2 dx + \int \frac{1}{y} dy = c$$

$$= -\frac{1}{y^3}\left[\frac{x^3}{3}\right] + \log y = c$$

$$= -\frac{x^3}{3y^3} + \log y = c$$

is the req^rd sol^n.

Non exact: 03 I.F

27 February 2025 14:08

Non - Exact D.E $\longrightarrow$ Exact D.E

Non - Exact D.E $\rightarrow$ Integrating factor (I.F) $\rightarrow$ Exact D.E

Steps or working Rule!

1) compare with std form $M\,dx + N\,dy = 0$, if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then D.E is Non-Exact, find I.F

2) $I.F = \frac{1}{Mx - Ny}$; $Mx + Ny \neq 0$

   Now, $I.F \times (M\,dx + N\,dy) = 0$

   then, $M'\,dx + N'\,dy = 0$

3) follow the same steps from S-①, ② , if Exact $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$ then find Gen^rl Sol^n

4) Gen^rl Sol^n! $\underset{(y-\text{const})}{\int M'\,dx} + \underset{(x\text{-eliminated})}{\int N'\,dy} = C$

#8 Solve, $y(x^2y^2 + 2)\,dx + x(2 - 2x^2y^2)\,dy = 0$

Sol^n S-① compare with $M\,dx + N\,dy = 0$

$M = y(x^2y^2 + 2)$ $\qquad$ $N = x(2 - 2x^2y^2)$

$\frac{\partial M}{\partial y} = 3x^2y^2 + 2$ $\qquad$ $\frac{\partial N}{\partial x} = 2 - 6x^2y^2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Given D.E is Non-Exact $\xrightarrow{I.F = \frac{1}{Mx - Ny}}$ Exact

$\overset{\downarrow}{\text{Exact}}$ $Mx - Ny$

S-2 $I.F = \dfrac{1}{Mx - Ny} = \dfrac{1}{(y(x^2y^2+2))x - (x(2-2x^2y^2))y}$

$$I.F = \frac{1}{3x^3y^3}$$

$I.F \times (M\,dx + N\,dy) = 0$

$$\Rightarrow \frac{1}{3x^3y^3} \times \left[y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy \right] = 0$$

$$\Rightarrow \frac{\cancel{y}(x^2y^2+2)}{3x^3y^{\cancel{3}}}\,dx + \frac{\cancel{x}(2-2x^2y^2)}{3x^{\cancel{3}}y^3}\,dy = 0$$

$$= \left[\frac{x^2y^2+2}{3x^3y^2}\right]dx + \left[\frac{2-2x^2y^2}{3x^2y^3}\right]dy = 0$$

$$= \left[\frac{1}{3x} + \frac{2}{3x^3y^2}\right]dx + \left[\frac{2}{3x^2y^3} - \frac{2}{3y}\right]dy = 0$$

Now $M'\,dx + N'\,dy = 0 \longrightarrow$ solve this find whether Exact?

$\therefore$ Gen'l sol'n : $\underset{(y\text{-const})}{\int M\,dx} + \underset{(x\text{-eliminated})}{\int N\,dy} = 0$

$N' = \left[\dfrac{2}{3x^2y^3} - \dfrac{2}{3y}\right]$

$$= \underset{y\text{-const}}{\int}\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \int \frac{-2}{3y}\,dy = c$$

$\because \int \frac{1}{y}\,dy = \log y$

$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \dfrac{x^{-3+1}}{-3+1} = -x^{-2}$

$$= \frac{1}{3}\int \frac{1}{x}\,dx + \frac{2}{3y^2}\int \frac{1}{x^3}\,dx - \frac{2}{3}\int \frac{1}{y}\,dy = c$$

$$= \frac{1}{3}\log x + \frac{\cancel{2}}{3y^2}\left[-\frac{\cancel{1}}{\cancel{2}x^2}\right] - \frac{2}{3}\log y = c$$

$$= \frac{1}{3}\log x + \frac{\cancel{2}}{3y^2}\left[-\frac{1}{\cancel{2}x^2}\right] - \frac{2}{3}\log y = C$$

$$-3+1$$

$$= -\frac{x^{-2}}{2}$$

$$= \frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = C$$ is the reqrd soln //

Non exact : 04th I.F

28 February 2025 09:53

Non - Exact D.E $\longrightarrow$ Exact D.E

Integrating factor (I.F)

Steps or working Rule!

1) Compare with std form $M\,dx + N\,dy = 0$, if $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ then D.E is Non-Exact, find I.F

2) ** $I.F = e^{\int f(x)\,dx}$; where ** $f(x) = \dfrac{1}{N}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right]$

   Now, ** $I.F \times (M\,dx + N\,dy) = 0$

   then, $M'\,dx + N'\,dy = 0$

3) follow the same steps from S-1, 2, if Exact $\dfrac{\partial M'}{\partial y} = \dfrac{\partial N'}{\partial x}$ then find Gen^rl Sol^n

4) Gen^rl Sol^n : ** $\underset{(y\text{-const})}{\int M'\,dx} + \underset{(x\text{-eliminated})}{\int N'\,dy} = C$

Q. $2xy\,dy - (x^2+y^2+1)\,dx = 0$

Sol : S-1 Compare with $M\,dx + N\,dy = 0$

$M = -(x^2+y^2+1)$ $\qquad\qquad$ $N = 2xy$

$\dfrac{\partial M}{\partial y} = -2y$ $\qquad\qquad$ $\dfrac{\partial N}{\partial x} = 2y$

$\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ $\therefore$ Given D-Eq^n is Non-Exact

$\therefore e^{\int f(x)\,dx} = I.F$ $\downarrow$ Exact

$f(x) = \dfrac{1}{N}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right]$

$$\overbrace{f(x)=\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]}^{\text{f C}}$$

Exact

S-② Since it is Non-Exact, Now find I.F

$$f(x)=\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=\frac{1}{2xy}\left[-2y-2y\right]=\frac{-4y}{2xy}$$

$$f(x)=\frac{-2}{x}$$

$$I.F=e^{\int f(x)\,dx}=e^{\int -\frac{2}{x}\,dx}=e^{-2\int \frac{1}{x}\,dx}$$

$$I.F=e^{-2\log x}$$

$$=e^{\log x^{-2}}$$

$$\left\{\because \int \frac{1}{x}\,dx=\log x \qquad \because e^{\log_e x}=x\right\}$$

$$I.F=x^{-2}=\frac{1}{x^2}$$

Now, $I.F\times(M\,dx+N\,dy)=0$

$$\Rightarrow \frac{1}{x^2}\times\left[-(x^2+y^2+1)\,dx+2xy\,dy\right]=0$$

$$\Rightarrow -\frac{(x^2+y^2+1)}{x^2}\,dx+\frac{2xy}{x^2}\,dy=0$$

$$\Rightarrow -\left[1+\frac{y^2}{x^2}+\frac{1}{x^2}\right]dx+\frac{2y}{x}\,dy=0$$

S-3: Now, M', N' values obtained, check for Exactness, $\frac{\partial M'}{\partial y}$, $\frac{\partial N'}{\partial x}$

$$\frac{\partial}{\partial y} \quad \frac{\partial}{\partial x}$$

$$M' = -\left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right] \qquad N' = \frac{2y}{x}$$

$$\frac{\partial M'}{\partial y} = -\frac{2y}{x^2} \quad ; \quad \frac{\partial N'}{\partial x} = -\frac{2y}{x^2}$$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$; Obtained D.E is Exact $\rightarrow$ Gen^rl sol^n

S-④: find Gen^rl Sol^n: $\underset{(y\text{-const})}{\int M'\,dx} + \underset{(x\text{-eliminated})}{\int N'\,dy} = C$

$$N' = \frac{2y}{x}$$

$$= \int_{y\text{-const}} \left[-\left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right]\right] dx + \int 0\,dy = c$$

$$= -\int 1\,dx - y^2 \int \frac{1}{x^2}\,dx - \int \frac{1}{x^2}\,dx = c$$

$$= -x - y^2\left[-\frac{1}{x}\right] - \left[-\frac{1}{x}\right] = c$$

$$= -x + \frac{y^2}{x} + \frac{1}{x} = c$$

$$= -x^2 + y^2 + 1 = xc$$ is the req^rd Sol^n

LINEAR DIFFERENTIAL EQN

05 March 2025 09:37

S-① : std form of L.D.E in term of 'y'

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

identify $P(x), Q(x)$

S-② find Integrating factor $= e^{\int P(x)\,dx}$

S-③ find Gen^rl Sol^n: $y[I.F] = \int [Q(x)\cdot(I.F)]\,dx + c$

you can find, L.D.E
interms of 'x'
By replace
$y \to x,\ x \to y$

#Q: Solve $x\frac{dy}{dx} + y = \log x$ —①

Sol S-① Divide eq① with 'x' on B.S,,

$$\frac{dy}{dx} + \frac{1}{x}\cdot y = \frac{\log x}{x} \longrightarrow ②$$

eq② ⇒ Compare with std form of L.D.E interm of 'y'

$$\frac{dy}{dx} + P(x)\cdot y = Q(x) \quad —③$$

$P(x) = \frac{1}{x}$, $Q(x) = \frac{\log x}{x}$

S-② find $I.F = e^{\int P(x)\,dx}$

$$= e^{\int \frac{1}{x}\,dx} = e^{\log x}$$

$\because \int \frac{1}{x}\,dx = \log x$

$\because e^{\log_e a} = a$

$I.F = x$

S-③ find Gen^rl Sol^n $y[I.F] = \int [Q(x)(I.F)]\,dx + c$

$$\Rightarrow y[x] = \int \left[\frac{\log x}{x} \times x\right] dx + c$$

$$y(x) = \int \log x\,dx + c$$

$$y(x) = x[\log x - 1] + c$$ is the reqr^d sol^n

$\because \int \log x\,dx = \int 1.\log x$
$= \int x^0 \log x$

Int by Parts

I
L → $\log x = u$
A — $x^0 = v$
T
E

$\int u.v\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$

E

$$\int u \cdot v \, dx = u \int v \, dx - \int \left(u' \int v \, dx \right) dx$$

$$\int \log x \cdot 1 \, dx = \log x \, (x) - \int \left(\frac{1}{\cancel{x}} \times \cancel{x} \right) dx$$

$$= x \log x - x$$

$$\boxed{\int \log x \, dx = x [\log x - 1]}$$

Equations reducible to L.D.E (Bernoulli's D.E)

05 March 2025 10:12

Steps: 01 Compare with std form: $\frac{dy}{dx} + Py = Q \cdot y^n$, identify P, Q

Step: 02: bring N.L.D.E eqn → L.D.E, find 1) I.F, 2) Gen'l soln

*Q. Solve $\frac{dy}{dx} + x \cdot \sin 2y = x^3 \cdot \cos^2 y$

Sol: Divide with $\cos^2 y$ on B.S

$\because \sin 2\theta = 2\sin\theta \cdot \cos\theta$

$\frac{1}{\cos\theta} = \sec\theta$

$\frac{\sin\theta}{\cos\theta} = \tan\theta$

$$\Rightarrow \frac{1}{\cos^2 y}\frac{dy}{dx} + x \cdot \frac{2\sin y \cancel{\cos y}}{\cos^{\cancel{2}} y} = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x \cdot \tan y = x^3 \quad \text{— (1)}$$

Put $K = \tan y$ — (2)

→ Diff 'K' w.r.t 'x'

$$\frac{dK}{dx} = \sec^2 y \times \frac{dy}{dx} \quad \text{— (3)}$$

Replace (2), (3) in (1)

$$\Rightarrow \frac{dK}{dx} + 2x \cdot K = x^3 \quad \text{— (4)}$$

Eq-(4) rep L.D.E in term of 'K'

$$\frac{dK}{dx} + P(x) K = Q(x) \quad \text{— (5)}$$

Compare & identify, $P(x) = 2x$, $Q(x) = x^3$

$\int x \, dx = \frac{x^2}{2}$

Now, $I.F = e^{\int P(x) dx} = e^{\int 2x \, dx} = e^{x^2}$

Now, Gen'l soln: $K[I.F] = \int [Q(x) \times (I.F)] \, dx + c$

Now, Gen^r^l sol^n : $K[I.F] = \int [Q(x) \times (I.F)]\, dx + c$

$\Rightarrow K[e^{x^2}] = \int x^3 \cdot e^{x^2}\, dx + c$

** By Parts ** Substitution

By Substitution

$\int x^3 e^{x^2} dx =$ | Put $x^2 = t$

$t = x^2$ — ①

$\Downarrow$

Diff 't' w.r.t 'x'

$\Rightarrow \int e^{x^2} x^2 \cdot x \cdot dx$

$\frac{dt}{dx} = 2x$

$dt = 2x\, dx$

$\Rightarrow \int e^t \cdot t \cdot \frac{dt}{2}$

$\frac{dt}{2} = x\, dx$ — ②

$\Rightarrow \frac{1}{2} \int e^t \cdot t\, dt$

$\hookrightarrow \int u \cdot v\, dx = u \int v\, dx - \int \left(u' \int v\, dx\right) dx$

ORTHOGONAL TRAJECTORIES

06 March 2025 12:08

find the O.T of family of curves: $y = kx$, k is a parameter.

NOTE: 1) Diff w.r.t 'x' ($y' = \frac{dy}{dx}$). (eliminate the Parameter)

2) replace $y' = -\frac{1}{y'}$ and seperate the Variable and Integrate.

Sol: Given : $y = kx$ — ①

Diff 'y' w.r.t 'x'

$$\frac{dy}{dx} = k(1)$$

$$y' = \frac{dy}{dx} = k \quad — ②$$

replace 'k' in ①.

$$\Rightarrow y = kx$$

$$y = y'.x \quad \rightarrow ③$$

S-② replace $y' = -\frac{1}{y'}$ in eq ③

$$y = -\frac{1}{y'}.x$$

$$y' = -\frac{1}{y}.x \quad \Big| \quad \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow y\,dy = -x\,dx$$ (Variable's are Seperated)

$$\Rightarrow \int y\,dy = -\int x\,dx$$ (Integrate On B.S)

$\Rightarrow \quad \int y\,dy = -\int x\,dx$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} = C \; ; \quad \Rightarrow x^2 + y^2 = 2C$ (circle eq^n) ——— (5)

$\therefore$ eq-(5) rep the O.T of the family of curves.

Unit 02:

12 March 2025 10:05

✓ Sol^n of 2^{nd} & higher ord L. Homo Diff eq^n with constants

for Homogenious L.D.Eq^n:

finding the Sol^n: $Y_{complete} = Y_{\text{Complementory fun}^c} + Y_{\text{particular Integral}}$

for Homogenious D.Eq^n: (R.H.S = 0)

Sol^n: $Y_{complete} = Y_{\text{complementory fun}^c}$

for Non-Homo : (R.H.S $\neq$ 0)

$Y_{complete} = Y_{C.F} + Y_{\text{Particular Integral}}$.

Homogenious D.Eq^n of 2^{nd} order:

Roots = 2 Roots

Real + Diff	$Y_{C.F} = c_1 e^{(\ )x} + c_2 e^{(\ )x}$
repeated	$= (c_1 + c_2 x) e^{(\ )x}$
Complex	$= e^{(\ )x}[c_1 \cos(\ )x + c_2 \sin(\ )x$

Ex: $\boxed{3, 5}$ ⇒ R & diff

$$Y_{C.F} = c_1 e^{(3)x} + c_2 e^{(5)x}$$

4, 4 = Repeated

$$Y_{C.F} = (c_1 + c_2 x) e^{(4)x}$$

$3+4i \Rightarrow$ Complex Value

Real val

$$Y_{C.F} = e^{(3)x}\left[C_1 \cos(4)x + C_2 \sin(4)x\right]$$

img val

$y'' + 6y' + 9y = 0, \quad y(0) = -4, \quad y'(0) = 14$

Sol D-oprtr $\Rightarrow (D^2 + 6D + 9)y = 0$

A-Eq = $m^2 + 6m + 9 = 0$ $9 - 3\times3$

$+3m + 3m$

$(m+3)(m+3) = 0$

$m = -3, -3$

$\{m = -3, -3\}$

$\therefore -3, -3 =$ repeated roots

$$Y_{C.F} = (C_1 + C_2 x)e^{-3x}$$

$$Y(x) = (C_1 + C_2 x)e^{-3x} \qquad y(0) = -4$$

$Y(x) = -4$

$$-4 = (C_1 + C_2(0))e^{-3(0)}$$

$$-4 = C_1 + 0$$

$$\boxed{C_1 = -4}$$

$$Y(x) = (C_1 + C_2 x)e^{-3x} \qquad y'(0) = 14$$

$$Y'(x) = C_1[-3.e^{-3x}] + C_2[x[-3e^{-3x}] + e^{-3x}[1]]$$

$$y'(x) = -3C_1 e^{-3x} - 3C_2 x e^{-3x} + C_2 e^{-3x}$$

$e^0 = 1$

$e^{-0} = 1$

$x=0$

$$14 = -3c_1 e^{-0} - 3c_2(0)(\ ) + c_2 e^{-0}$$

$\{e^{-0}\} = \frac{1}{e^0} = \frac{1}{1}$

$= 1$

$$14 = -3c_1 + c_2 \qquad \because c_1 = -4$$

$$14 = -3(-4) + c_2$$

$$c_2 = 14 - 12 = 2$$

$$\therefore c_1 = -4, \quad c_2 = 2$$

$$\therefore y(x) = (c_1 + c_2 x) e^{-3x}$$

$$= (-4 + 2x) e^{-3x}$$

#: Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

Sol A-Eqn: $m^4 - 2m^3 - 3m^2 + 4m + 4 = 0$

$\therefore$ Roots are: $m = 2, -1, -1, 2$

$\therefore$ $-1, -1$ = Same & Repeated, ① ② $\qquad$ $2, 2$ – Same & repeated,

$$y_{c.F} = (c_1 + c_2 x) e^{-x} + (c_3 + c_4 x) e^{2x}$$

Solution of non homogeneous linear differential equations with constant coefficients

19 March 2025 09:35

Quick review :

L.D.E with higher order $\longrightarrow$ Unit:02

$\rightarrow$ Tot sol = C.Fun^c + Particular Int^{gral}

R.H.S = 0 | R.H.S $\neq$ 0

Homogeneous L.D.E of H.ord | Non-Homo L.D.E of H.ord } 6m

Tot sol^n = C.Fun^c | Tot sol^n = C.Fun^c + Particular Int^{gl} — C.F + P.I //

8m

D.E $\rightarrow$ D-opr^{rt} $\rightarrow$ A.Eq^n $\rightarrow$ ~~R.H.S~~

$D \rightarrow m$ | y

roots

Formulas Select: C.F,

If initial Cond: $y(0) = K_1$

$y'(0) = K_2$

R.H.S $\neq$ 0

1. e^{ax}
2. $\sin ax$ (or) $\cos ax$
3. x^K

} SAQ

4. Comb: $e^{ax} \cdot V$ — $\sin ax$ (or) $\cos ax$, x^K
5. $x^K \cdot V$ — e^{ax}, $\sin ax$ (or) $\cos ax$

★★ LAQ

★★

Type:01 R.H.S = e^{ax} then

$$f(D) \cdot y_p = e^{ax}$$

$$y_p = \frac{1}{f(D)} \cdot e^{ax} \Big|_{D=a}$$

$\therefore$ $y = y_{c.F} + y_p$ // Tot sol^n :

#1. Solve, $(D^3 - D^2 - 4D + 4)y = e^{3x}$

Sol: Auxillary eqⁿ: $f(m)=0$, $m^3-m^2-4m+4=0$

Roots: are $m = 1, 2, -2$ $\therefore$ roots are real & distinct

$$y_{c.F} = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-2x}$$

Now Particular Integral: $f(D)\, y_p = e^{3x}$

$$\Rightarrow y_p = \frac{1}{f(D)} \cdot e^{3x} \Big|_{D=3}$$

$$= \frac{1}{D^3 - D^2 - 4D + 4} \cdot e^{3x} \Big|_{D=3}$$

$$= \frac{1}{(3)^3 - (3)^2 - 4(3) + 4} \cdot e^{3x}$$

$$= \frac{1}{27 - 9 - 12 + 4} \cdot e^{3x}$$

$$y_{p.I} = \frac{1}{10} \cdot e^{3x}$$

$$\therefore y = y_{c.F} + y_{p.I}$$

$$= c_1 e^{x} + c_2 e^{2x} + c_3 e^{-2x} + \frac{e^{3x}}{10}$$

#3. Solve $(D^2 + 2D + 1)y = e^{-x}$

Sol A-Eqⁿ: $m^2 + 2m + 1 = 0$

Roots: of m are $= -1, -1$ $\therefore$ roots are repeated

$$y_{c.F} = (c_1 + c_2 x) e^{-x}$$

Now, Particular Integral: y_p:

$f(D) y = e^{-x}$

Now, particular integral of:

$$f(D)\,y_p = e^{-x}$$

$$y_p = \frac{1}{f(D)}\cdot e^{-x} = \frac{1}{(D^2+2D+1)}\cdot e^{-x}\Big|_{D=a=-1}$$

$$\therefore y_p = \frac{1}{(-1)^2+2(-1)+1}\cdot e^{-x}$$

$$= \frac{1}{1-2+1}\cdot e^{-x} =$$

S/c

$$\therefore y_p = \frac{x}{2D+2}\cdot e^{-x}\Big|_{D=-1} = \frac{x}{2(-1)+2}\cdot e^{-x} =$$

$$= \frac{x\;x}{2}\,e^{-x}$$

$$y_p = \frac{x^2}{2}\cdot e^{-x}$$

$$\therefore y = y_c + y_p = (c_1+c_2x)\,e^{-x} + \frac{x^2}{2}\cdot e^{-x}$$

#6: Solve $(D^2+6D+9)y = 2e^{-3x}$

Sol: $\because$ $y_c = (c_1+c_2x)\,e^{-3x}$

Now, P.I.;

$$y_p = \frac{1}{f(D)}\,2e^{-3x} = 2\frac{1}{\boxed{D^2+6D+9}}\,e^{-3x}\Big|_{D=-3}$$

$$\Rightarrow y_p = 2\cdot\frac{1}{}\cdot e^{-3x}$$

$$\Rightarrow \quad y_p = 2 \cdot \frac{1}{(-3)^2 + 6(-3) + 9} \cdot e^{-3x}$$

$$= 2 \cdot \frac{1}{9 - 18 + 9} \cdot e^{-3x} \qquad \underline{\underline{Dr = 0}}$$

$$= 2 \cdot \frac{x}{2D + 6} \cdot e^{-3x} \qquad \Bigg| \; D = -3$$

$$= 2 \; \frac{x}{2(-3) + 6} \cdot e^{-3x}$$

$$= 2 \; \frac{x \cdot x}{2} \; e^{-3x}$$

$$y_p = 2 \cdot \frac{x^2}{2} \cdot e^{-3x}$$

$$y_p = x^2 \cdot e^{-3x}$$

$$\therefore \; y = y_c + y_p = (c_1 + c_2 x) e^{-3x} + x^2 e^{-3x} \; //$$

Type 02 sin ax or cos ax

22 March 2025 11:53

$$f(D^2)y = \sin ax \text{ (or) } \cos ax$$

$$y_p = \frac{1}{f(D^2)} \cdot \sin ax \text{ (or) } \cos ax \quad \Big| \; D^2 = -a^2$$

#1: Solve $(D^2 - 4D + 3)y = \cos 2x$

Sol

Roots: $3, 1$ roots are real & distinct.

C.F: $y_{C.F} = c_1 e^{3x} + c_2 e^{x}$

P.I: $f(D) \cdot y_p = \cos 2x$

$$y_p = \frac{1}{f(D)} \cdot \cos 2x \quad \Big| \; D^2 = -a^2$$

$$\Rightarrow y_p = \frac{1}{D^2 - 4D + 3} \cdot \cos 2x \quad \Big| \; D^2 = -(2^2) = -4$$

$$= \frac{1}{-4 - 4D + 3} \cdot \cos 2x$$

$$= \frac{1}{-4D - 1} \cdot \cos 2x$$

Rationalize the Dr.

$$= \frac{1}{-4D - 1} \times \frac{-4D + 1}{-4D + 1} \times \cos 2x$$

$(a - b)(a + b) = a^2 - b^2$

$$= \frac{(-4D + 1)\cos 2x}{(-4D - 1)(-4D + 1)}$$

$$= \frac{(-4D + 1)\cos 2x}{(-4D)^2 - (1)^2}$$

$(-2)^2 x = 4x$

$$= \frac{(-4D + 1)\cos 2x}{16D^2 - 1} \quad \Big| \; D^2 = -a^2 = -(2)^2 = -4$$

$$= \frac{(-4D + 1)\cos 2x}{16(-4) - 1}$$

$$= (-4D + 1)\cos 2x$$

$D = \frac{d}{dx}$

$16(-4)-1$

$$= \frac{(-4D+1)\cos 2x}{-65}$$

$D = \frac{d}{dx}$

$$= \frac{-4\frac{d}{dx}(\cos 2x) + \cos 2x}{-65}$$

$\frac{d}{dx}(\cos 2x) =$

$= -\sin(2x) \times 2$

$= -2\sin 2x$

$$= \frac{-4(-2\sin 2x) + \cos 2x}{-65}$$

$$y_p = -\frac{8\sin 2x}{65} - \frac{\cos 2x}{65}$$

$\therefore y = y_c + y_p$

$$= c_1 e^{3x} + c_2 e^{x} - \frac{8\sin 2x}{65} - \frac{\cos 2x}{65} \quad //$$

#3 : Solve $(D^2 - 4D + 3)y = \sin 3x \cdot \cos 2x$

Sol

$\therefore$ Roots: 3, 1, $\therefore$ roots are R + Dist

CF: $y_{C.F} = c_1 e^{3x} + c_2 e^{x}$ //

PI:

$f(D^2) \cdot y_p = \sin 3x \cdot \cos 2x$

$\because \sin A \cos B$

$(D^2 - 4D + 3) y_p = \frac{1}{2} \cdot [\sin(5x) + \sin x]$

$= \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$$\Rightarrow y_p = \frac{1/2 \sin 5x}{D^2 - 4D + 3} + \frac{1/2 \sin x}{D^2 - 4D + 3}$$

$$= y_{p_1} + y_{p_2}$$

$$\therefore y_{p_1} = \frac{1/2 \sin 5x}{D^2 - 4D + 3}$$

$D^2 = -a^2 = -(5)^2 = -25$

$$= \frac{1/2 \sin 5x}{-25 - 4D + 3}$$

$$= \frac{\frac{1}{2}\sin 5x}{-4D-22}$$

Rationalize the Dr:

$$= \frac{\frac{1}{2}\sin 5x}{-4D-22} \times \frac{-4D+22}{-4D+22}$$

$$= \frac{\frac{1}{2}\sin 5x\,(-4D+22)}{(-4D-22)(-4D+22)}$$

$(a-b)(a+b) = a^2-b^2$

$$= \frac{\frac{1}{2}\cdot\sin 5x\,(-4D+22)}{(-4D)^2-(22)^2}$$

$22 \times 22 = 44 + 44\,\, = 484$

$$= \frac{\frac{1}{2}\sin 5x\,(-4D+22)}{16D^2-484}$$

$D^2 = -a^2 = -25$

$$= \frac{\frac{1}{2}\sin 5x\,(-4D+22)}{16(-25)-484}$$

$16 \times 25 = 400$

$$= \frac{\frac{1}{2}\sin 5x\,(-4D+22)}{-884}$$

$$= \frac{\frac{1}{2}\sin 5x\,(-4D+22)}{-884}$$

$D = \frac{d}{dx}$

$$= \frac{-4\frac{d}{dx}\left(\frac{1}{2}\sin 5x\right) + \overset{11}{\cancel{22}}\left[\frac{1}{\cancel{2}}\sin 5x\right]}{-884}$$

$\frac{d}{dx}(\sin 5x) = \cos(5x)\times 5 = 5\cos 5x$

$$= \frac{-\overset{2}{\cancel{4}}\times\frac{1}{\cancel{2}}\left[5\cos 5x\right] + 11\sin 5x}{-884}$$

$$= \frac{-10\cos 5x + 11\sin 5x}{-884}$$

$$y_p = \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x$$

$$y_{p_1} = \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x$$

$$y_{p_2} = \frac{\frac{1}{2}\sin x}{D^2 - 4D + 3} \quad \Big| \; D^2 = -a^2 = -1$$

$$= \frac{\frac{1}{2}\sin x}{-1 - 4D + 3} = \frac{\frac{1}{2}\sin x}{-4D + 2}$$ Rationalize the 'Dr'

$$= \frac{\frac{1}{2}\sin x}{-4D + 2} \times \frac{-4D - 2}{-4D - 2} \quad [\because (a-b)(a+b) = a^2 - b^2]$$

$$= \frac{\frac{1}{2}\sin x\,(-4D - 2)}{16D^2 - 4} \quad \Big| \; D^2 = -a^2 = -1$$

$$= \frac{\frac{1}{2}\sin x\,(-4D - 2)}{-20}$$

$$\left\{ \begin{array}{l} \because D = \frac{d}{dx} \\ \because \frac{d}{dx}(\sin x) = \cos x \end{array} \right.$$

$$= \frac{-4\left(\frac{d}{dx}\left(\frac{1}{2}\sin x\right)\right) - 2\left(\frac{1}{2}\sin x\right)}{-20}$$

$$y_{p_2} = \frac{-2\cdot\cos x - \sin x}{-20} = \frac{\cos x}{10} + \frac{\sin x}{20}$$

$$\therefore y_p = y_{p_1} + y_{p_2}$$

$$y_p = \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x + \frac{\cos x}{10} + \frac{\sin x}{20}$$

$$\therefore y = y_c + y_p =$$

$$= c_1 e^{3x} + c_2 e^{x} + \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x + \frac{\cos x}{10} + \frac{\sin x}{20}$$

Quick review + type 03

02 April 2025 09:41

U-2

1.

R.H.S = 0 → H.Eqn
- → D-op
  - ↓ A.Eqn ($D \to m$, ~~y~~, ~~R.H.S~~)
  - ↓ Roots
  - ↓ $\boxed{C.F}$

R.H.S ≠ 0 → N.H.Eqn
- → 1) e^{ax} (exp) | $D = a$
- 2) $\sin ax$ (or) $\cos ax$ | $D^2 = -a^2$
- 3) x^k (algeb) |
- 4) $e^{ax} \cdot v$
- 5) $x^k v$

$$y_p = \frac{1}{f(D)}(\text{R.H.S})$$

Note:-

1) $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$
2) $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$
3) $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$
4) $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$
5) $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots$
6) $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \cdots$

ii) $(D^3+2D^2+D)y = x$

\# 1. i) Solve $(D^2+D+1)y = x^3$

Sol into Auxillary eqn: $f(m) = 0$ | $D \to m$, $f(D)\,y = x^3$ ($D = m$)

$\Rightarrow$ $m^2 + m + 1 = 0$

Roots: are $\frac{-1 \pm i\sqrt{3}}{2}$ are complex roots

C.F, $y_{c.f} = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\right)x + c_2 \sin\left(\frac{\sqrt{3}}{2}\right)x\right]$

Now, Particular Integral,

Now, Particular Integral,

$$f(D)\,y_p = x^3$$

$$\Rightarrow y_p = \frac{1}{f(D)}\cdot x^3$$

$$= \frac{1}{D^2+D+1}\cdot x^3$$

$$[1 \pm \phi(D)]^{\pm 1}$$

$$D^2+D+1 = 1+(D^2+D)$$

$$(1+\phi(D))^{-1}$$

$$\phi(D) = D^2+D$$

$$y_p = \frac{1}{[1+(D^2+D)]}\cdot x^3$$

$$y_p = [1+(D^2+D)]^{-1}\cdot x^3$$

$$[1+D]^{-1} = 1 - D + D^2 - D^3 + D^4 \ldots\ldots$$

$$\Rightarrow y_p = [1-(D^2+D)+(D^2+D)^2-(D^2+D)^3+\ldots\ldots]\,x^3$$

$$(D^2+D)^2 = [D^4+D^2+2D^3]\,x^3$$

$$y_p = [1-(D^2+D)+(D^2+D)^2]\cdot x^3$$

$$D^4(x^3)$$

$$= 3x^2 \quad - D$$

$$6x \quad - D^2$$

$$6 \quad - \boxed{D^3}$$

$$0 \quad - D^4$$

$$= [x^3 - 6x - 3x^2 + (0 + 6x + 2(6))]$$

$$= x^3 - \cancel{6x} - 3x^2 + \cancel{6x} + 6$$

$$y_p = x^3 - 3x^2 + 12$$

$$\therefore y = y_{C.F} + y_p$$

$$y = e^{-1/2x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right] + x^3 - 3x^2 + 12$$

from Question Bank

#15. Solve, $\dfrac{D^2y}{dx^2} - 5y = 5x - 2$

$$[1 \pm \phi(D)]^{\pm 1}$$

Sol

$$y_p = \frac{1}{D^2-1}\cdot(5x-2)$$

$$\frac{1}{D^2-1}$$

$$= \frac{1}{-[1-D^2]}(5x-2)$$

$$y_p = -[1-D^2]^{-1}(5x-2)$$

$[1-D]^{-1} = 1+D+D^2+D^3+\ldots$

$$y_p = -[1+D^2+D^4+D^6+\ldots\ldots](5x-2)$$

$5x-2$

$D \to 5$

$D^2 \to 0$

$$= -[5x-2]$$

$$y_p = -5x+2$$

$x^2 + 1$

$$\therefore y = y_c + y_p \;//$$

Type 04

02 April 2025 10:17

$$f(D)\cdot y = e^{ax}\cdot v \Big|_1$$

$\Rightarrow$ Particular Integral, $y_p = \dfrac{1}{f(D)}\cdot e^{ax}\cdot v \Big|_{D\to D+a}$

$$= e^{ax}\cdot \frac{1}{f(D+a)}\cdot v$$

v: $\sin ax$ (or) $\cos ax$ $\Big|_{D^2=-a^2}$

$x^k \to$ Binomial exp.^{ns}

from Q.B

19: Solve the D.Eq^n: $(D^3+1)y = e^{2x}\cdot \sin x$

Sol: for complementary fun^c:

A.Eq^n: $f(m)=0$

$$m^3+1=0$$

$\therefore$ Roots: $m = -1, \dfrac{1\pm i\sqrt{3}}{2}$

$\therefore$ C.F,

$$y_{C.F} = C_1 e^{-x} + e^{\frac{1}{2}x}\left[C_2\cos\left(\frac{\sqrt{3}}{2}\right)x + C_3\sin\frac{\sqrt{3}}{2}x\right]$$

P.I, $f(D)\cdot y_p = e^{2x}\cdot \sin x$

$$\Rightarrow y_p = \frac{1}{f(D)}\cdot e^{2x}\cdot\sin x\Big|_{D\to D+2}$$

$$= \frac{1}{D^3+1}\cdot e^{2x}\cdot \sin x\Big|_{D\to D+2}$$

$$= e^{2x}\frac{1}{(D+2)^3+1}\cdot\sin x$$

$(a+b)^3 =$

$D+2 = D^3+8+3D^2\times 2 + 3\times D\times 4$

$e^{2x}\ \ 1\ \ \ \sin x$

$$= e^{2x} \frac{1}{(D+2)^3+1} \cdot \sin x$$

$$(D+2)^3 = D^3 + 8 + 3 \times D^2 \times 2 + 3 \times D \times 4$$

$$= D^3 + 8 + 6D^2 + 12D$$

$$= e^{2x} \frac{1}{D^3 + 8 + 6D^2 + 12D + 1} \cdot \sin x \quad a=1$$

$$D^2 = -(1^2) = -1$$

$$= e^{2x} \cdot \frac{1}{-D + 8 + 6(-1) + 12D + 1} \cdot \sin x$$

$$= e^{2x} \cdot \frac{1}{11D + 3} \cdot \sin x$$

Rationalize the 'Dr'

$$= e^{2x} \cdot \frac{1}{11D+3} \times \frac{11D-3}{11D-3} \times \sin x$$

$$(a-b)(a+b) = a^2 - b^2$$

$$= e^{2x} \cdot \frac{(11D-3)}{121D^2 - 9} \cdot \sin x$$

$$D^2 = -1$$

$$= e^{2x} \cdot \frac{(11D-3)}{-121-9} \cdot \sin x$$

$$y_p = \frac{e^{2x}}{-130} \cdot \left(11 \frac{d}{dx}(\sin x) - 3 \sin x\right)$$

$$= -\frac{e^{2x}}{130} (11 \cos x - 3 \sin x)$$

$$\therefore y = y_c + y_p$$

Type 04 continue…..

03 April 2025 13:34

#20. from Question bank.

#20. Solve the differential equation: $\dfrac{d^2y}{dx^2} - 4y = x\sinh x$

Excercise the above Question. $\left\{\text{hint: } \sinh x = \dfrac{e^x - e^{-x}}{2}\right\}$

#2. Solve, $(D^2 - 6D + 13)y = 8e^{3x}\sin 2x$

Sol:

A.Eqⁿ: $f(m) = 0$, $m^2 - 6m + 13 = 0$

Roots: $m = 3 \pm 2i$ $\therefore$ roots are complex.

C.F: $y_{c.F} = e^{3x}[c_1\cos 2x + c_2\sin 2x]$ $\frac{1}{4}$ th Solⁿ

P.I: $f(D)y_p = 8e^{3x}\sin 2x$

$y_p = \dfrac{1}{f(D)} \cdot 8e^{3x}\sin 2x \Big|_{D \to D+3}$ std form: $y_p = \dfrac{1}{f(D)} e^{ax}\sin bx \Big|_{D \to D+a}$

$= \dfrac{1}{D^2 - 6D + 13} 8e^{3x}\sin 2x \Big|_{D \to D+3}$

$= 8e^{3x}\dfrac{1}{(D+3)^2 - 6(D+3) + 13}\sin 2x = 8e^{3x} \times \dfrac{1}{D^2 + 9 + 6D - 6D - 18 + 13}\sin 2x$

$= 8e^{3x} \times \dfrac{1}{D^2 + 4} \times \sin 2x \Big|$ $D^2 = -a^2 = -(4)$

$= 8e^{3x} \times \dfrac{1}{-4 + 4} \times \sin 2x$

$D^2 + 4$ $\qquad$ $\therefore D = -a = -(4)$

$= 8e^{3x} \times \dfrac{1}{-4+4} \times \sin 2x$

$\hookrightarrow f(-a^2) = 0,$

$= 8e^{3x} \times \dfrac{x}{2D} \times \sin 2x$ $\qquad$ $\dfrac{1}{D} \Rightarrow$ Int ; $D \to Dr \to$ Rationalize,,

$= 8e^{3x} \times \dfrac{x}{2} \cdot \int \sin 2x \, dx$ $\qquad$ $\left[\because \int \sin ax \, dx = \dfrac{-\cos ax}{a}\right.$

$= 8 \times e^{3x} \times \dfrac{x}{2}\left[-\dfrac{\cos 2x}{2}\right]$

$= -8e^{3x} \times \dfrac{x}{4} \times \cos 2x$

$y_p = -2e^{3x} \times x \times \cos 2x$,,

$\therefore y = y_c + y_p = e^{3x}[c_1 \cos 2x + c_2 \sin 2x] + (-2e^{3x} \times x \times \cos 2x)$,

Feb/mar - 2025 $\longrightarrow$ 6m

LAQ: 12 a) solve the D.Eq^n $(D^2+4D+3)y = e^x \cos 2x + 125$

Sol: D.eq to D-op^{rtr} $\Rightarrow f(D)y = e^x \cos 2x + 125$

A.Eq^n: $f(m) = 0$,
$m^2 + 4m + 3 = 0$

Roots: $m = -1, -3$ $\quad$ roots are real + distinct,, $\qquad$ $6 \times \frac{1}{4}$

C.F: $y_{c.f} = c_1 e^{-x} + c_2 e^{-3x}$ $\longrightarrow$ $1/4^{th}$ Sol^n

P.I: $f(D)y_p = e^x \cos 2x + 125\, e^{0x}$

$$\Rightarrow y_p = \frac{1}{f(D)} e^{x}\cos 2x + 125 \cdot \frac{1}{f(D)} e^{0x}$$

$$\Rightarrow y_p = \frac{1}{D^2+4D+3} e^{x}\cos 2x \Big|_{D\to D+1} + 125 \cdot \frac{1}{D^2+4D+3} e^{0x} \Big|_{D\to 0}$$

$$= e^{x} \cdot \frac{1}{(D+1)^2+4(D+1)+3} \cos 2x + 125 \frac{1}{3}$$

$$= e^{x} \frac{1}{D^2+1+2D+4D+7} \cos 2x + \frac{125}{3}$$

$$= e^{x} \frac{1}{D^2+6D+8} \cos 2x \Big|_{D^2=-a^2=-4} + \frac{125}{3}$$

$$= e^{x} \frac{1}{-4+6D+8} \cos 2x + \frac{125}{3}$$

$$= e^{x} \frac{1}{6D+4} \cos 2x + \frac{125}{3} \qquad \because \text{Rationalize the 'Dr'}$$

$$= e^{x} \frac{1}{6D+4} \times \frac{6D-4}{6D-4} \times \cos 2x + \frac{125}{3} \qquad \because (a+b)(a-b) = a^2-b^2$$

$$= e^{x} \frac{(6D-4)}{36D^2-16} \times \cos 2x \Big|_{D^2=-a^2=-4} + \frac{125}{3}$$

$$= e^{x} \frac{(6D-4)}{36(-4)-16} \times \cos 2x + \frac{125}{3}$$

$$= -\frac{e^{x}}{144}\left(6 \frac{d}{dx}(\cos 2x) - 4\cos 2x\right) + \frac{125}{3}$$

$$\because \frac{d}{dx}(\cos 2x) = -\sin 2x \times 2$$

$$= \frac{-e^{x}}{144}\left(-6(2\sin 2x) - 4\cos 2x\right) + \frac{125}{3}$$

$= \frac{-e^{x}}{144}(-6(2\sin 2x) \quad (\qquad) + \frac{\quad}{3}$ $= -\sin 2x \times 2$

$= -2\sin 2x$

$$y_p = \frac{-e^{x}}{144}(-12\sin 2x - 4\cos 2x) + \frac{125}{3} \Rightarrow$$

$$\therefore y = y_c + y_p \qquad \rightarrow \quad \frac{3}{4}^{th}\ \underline{\underline{sol^n}}$$

$$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{e^{x}}{144}(-12\sin 2x - 4\cos 2x) + \frac{125}{3} \quad //$$

Type 5

03 April 2025 14:22

$f(D)y = x\cdot V$ | $f(D)\cdot y = x^K \cdot V$ ↳ Later discussed

s/c

$\Rightarrow y_p = \left[x - \frac{f'(D)}{f(D)}\right] \times \frac{1}{f(D)} \times V$

\# Solve the D.Eqn: $(D^2 + 2D + 1)y = x\cos x$.

Sol D-optr: $f(D)\cdot y = x\cos x$

A.Eqn: $f(m) = 0$, $m^2 + 2m + 1 = 0$

Roots: $m = -1, -1$ $\therefore$ roots are repeated

C.F: $y_{C.F} = (c_1 + c_2 x)e^{-x}$

P.I: $f(D)\cdot y_p = x\cos x$

$\Rightarrow y_p = \frac{1}{f(D)}\cdot x\cos x$

Std form

$y_p = \frac{1}{f(D)}\cdot x \cdot V$

$= \left[x - \frac{f'(D)}{f(D)}\right] \times \frac{1}{f(D)} \times V$

$y_p = \frac{1}{D^2+2D+1} \times x\cos x$

$\Rightarrow y_p = \left[x - \frac{2D+2}{D^2+2D+1}\right] \times \frac{1}{D^2+2D+1} \times \cos x$

$= x \times \frac{1}{D^2+2D+1} \times \cos x \Big|_{D^2 = -a^2 = -1} - \frac{(2D+2)}{(D^2+2D+1)^2} \times \cos x \Big|_{D^2 = -a^2 = -1}$

$= x \times \frac{1}{2D} \times \cos x - \frac{(2D+2)}{(2D)^2} \times \cos x \Big|_{D^2 = -a^2 = -1}$

$$= \frac{x}{2} \times [\sin x] - \frac{(2D+2)}{4(-1)} \times \cos x$$

$$= \frac{x}{2}\sin x + \frac{1}{4}\left[2\frac{d}{dx}(\cos x) + 2\cos x\right]$$

$$= \frac{x}{2}\sin x + \frac{1}{2}(-\sin x) + \frac{1}{2}\cos x$$

$$y_p = \frac{x}{2}\sin x - \frac{\sin x}{2} + \frac{\cos x}{2}$$

$$\therefore y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^{-x} + \frac{x}{2}\sin x - \frac{\sin x}{2} + \frac{\cos x}{2} \quad //$$

Practice

#. Solve the D.Eq^n! $(D^2 + 3D + 2)y = e^{e^x}$

Method of variation of parameters

03 April 2025 15:04

→ Also another method To find Particular Integral is **MoVoP**

Working Rule $f(D)\cdot y = R$ (R.H.S)

1). first find C.F (Complementary func^c)

Ex: $= c_1\cos x + c_2 \sin x$

consider it as $= A\cos x + B\sin x$

2) Now find A, B constants

where
$$A = -\int \frac{v\cdot R}{|w|}, \quad B = \int \frac{uR}{|w|}$$

where
$$|w| = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \text{ (or) } u\frac{dv}{dx} - v\frac{du}{dx}$$

R = obtained from R.H.S,

∓ Solve the D.Eq^n $(D^2+1)y = \operatorname{cosec} x$ by method of variation of Parameters.

Sol: D.Eq → D.oprtr → A.Eq^n

$f(D)y = RHS$ ↳ $f(m) = 0$

$f(D)\cdot y = R$

A.Eq: $f(m) = 0$

$m^2+1 = 0$

Roots: $m = \pm i$ ∴ Roots are Complex, (Real Val $= 0$, Img Val $= 1$)

C.F: $y_{cF} = e^{0x}[c_1\cos x + c_2\sin x]$

$y = c_1\cos x + c_2\sin x$

By M.V.P

$y = A\cos x + B\sin x$

By Method of V.P

$$y = A\cos x + B\sin x$$

Now $\cdot = Au + Bv$ | $u = \cos x,\ v = \sin x$

when $A = -\int \frac{V \cdot R}{|W|} dx$, $B = \int \frac{u R}{|W|} dx$, R = R.H.S of given Question

$$|W| = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$|W| = 1$

$\operatorname{cosec} x = \frac{1}{\sin x}$

Now $A = -\int \frac{\sin x \times (\operatorname{cosec} x)}{1} dx$

$= -\int \sin x \times \frac{1}{\sin x} dx =$

$\int 1 dx = x$

$A = -x$

$B = \int \frac{u R}{|W|} dx = \int \frac{\cos x \times \operatorname{cosec} x}{1} dx$

$= \int \cot x\, dx = \log|\sin x|$

$\int \cot x\, dx = \log|\sin x| + c$

$\therefore y = A\cos x + B\sin x$; $A = -x$

$B = \log|\sin x|$

$y = -x\cos x + \log|\sin x| \cdot \sin x$

Cauchy' s Euler equations.

07 April 2025 10:54

$x^2 y''$ $\quad x y'$ | $x^3 y'''$ $\quad x^2 y''$

W.K

1) Let $x = e^t$, $t = \log x$

2) $x^2 D^2$ (or) $x^2 y'' = \theta(\theta - 1)$; $x^3 D^3$ (or) $x^3 y''' = \theta(\theta-1)(\theta-2)$

xD (or) $xy' = \theta$; $x^4 D^4 = \theta(\theta-1)(\theta-2)(\theta-3)$

#. Solve $3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x$ —— ①

Sol Let i) $x = e^t$
$t = \log x$ } —— ②

2) $x^2 D^2 = x^2 y'' = \theta(\theta-1)$
$xD = xy' = \theta$ } —— ③

Sub ②,③ in ①,

$= (3x^2D^2 + xD + 1) = x$

$= 3[\theta(\theta-1)] + \theta + 1 = x$

$= 3\theta^2 - 3\theta + \theta + 1 = x$

$\Rightarrow 3\theta^2 - 2\theta + 1 = x$

$f(\theta) y = x$

↳ A-Eqn: $f(m) = 0$

$3m^2 - 2m + 1 = 0$

Roots:

$3m^2 - 2m + 1 = 0$

Roots:

$\hookrightarrow \quad m = \frac{1 \pm i\sqrt{2}}{3} = \therefore$ roots are Complex.

C.F: $y_c = e^{\frac{1}{3}x}\left[c_1 \cos\frac{\sqrt{2}}{3}x + c_2 \sin\frac{\sqrt{2}}{3}x\right]$,

P.I: $y_p = ;\quad f(\theta)\, y_p = x$

$\Rightarrow y_p = \frac{1}{f(\theta)} \cdot x$

$= \frac{1}{3\theta^2 - 2\theta + 1} \cdot x$

$f(\theta) = 1 \pm \phi(\theta)$

$3\theta^2 - 2\theta + 1 = 1 + (3\theta^2 - 2\theta)$

$y_p = \frac{1}{1 + (3\theta^2 - 2\theta)} \cdot x$

$= \left[1 + (3\theta^2 - 2\theta)\right]^{-1} \cdot x$

$\because [1 + D]^{-1} = 1 - D + D^2 - D^3 + D^4 \ldots$

$y_p = \left[1 - (3\theta^2 - 2\theta) + (3\theta^2 - 2\theta)^2 - (3\theta^2 - 2\theta)^3 + \cdots\right] x$

x

$D \longrightarrow 1$

$D^2 \longrightarrow 0$

$\therefore \boxed{y_p = x}$

$\therefore y = y_c + y_p$

$= e^{\frac{1}{3}x}\left[c_1 \cos\frac{\sqrt{2}}{3}x + c_2 \sin\frac{\sqrt{2}}{3}x\right] + x$ //

$x = e^t$

Comparison test

Basic formulae:

1) $\lim_{n\to\infty} \frac{1}{n} = 0$

2) $\lim_{n\to\infty} \frac{1}{n^2} = 0$

3) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

4) $\lim_{n\to\infty} \left[1+\frac{1}{n}\right]^n = e$

Test for the convergence of $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots$

Sol. " find the n^{th} Term.

Nr: $1, 3, 5, \ldots$; $a = 1$, $d = 2$

$n^{th} = a + (n-1)d = 1 + (n-1)2 = 2n-2+1$
$= 2n-1$

Nr: $n^{th} = 2n-1$

Dr:
series-01: $1, 2, 3 \ldots$; $a = 1$, $d = 1$

$n^{th} = a + (n-1)d = 1 + (n-1)1$
$= 1 + n - 1 = n$

$n^{th} = n$

series: 02, $2, 3, 4 \ldots$; $a = 2$, $d = 1$

n^{th}: $a + (n-1)d = 2 + (n-1)1$
$= 2 + n - 1 = n+1$

$n^{th} = n+1$

Series: 03, $3, 4, 5 \ldots$; $a = 3$, $d = 1$

n^{th}: $a + (n-1)d = 3 + (n-1)1$
$= 3 + n - 1$
$= n+2$

$n^{th} = n+2$

$= n+2$

$n^{th} = n+2$

Now! n^{th} Term of the Series: $= \dfrac{2n-1}{n(n+1)(n+2)} = U_n$

By Comparision Test.

$$\Sigma V_n = \frac{n}{n \cdot n \cdot n} = \frac{1}{n^2}$$

$\bigstar\bigstar$ $\Sigma V_n = \dfrac{1}{n^p}$

$\therefore$ "By Auxillary P-Test",

$p = 2$;

$\therefore$ $\therefore$ if $p > 1$ $\therefore$ ΣV_n is convergent,,

$2 > 1$

$\bigstar\bigstar$ $\therefore \Sigma V_n$ is convergent,,

$$\boxed{\therefore \lim_{n\to\infty} \frac{U_n}{V_n}} = \lim_{n\to\infty} \frac{\dfrac{2n-1}{n(n+1)(n+2)}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{\dfrac{n[2-1/n]}{n.n\ n\ (1+1/n)(1+2/n)}}{\dfrac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{2-1/n}{(1+1/n)(1+2/n)} = \frac{(2-0)}{(1+0)(1+0)}$$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$$= \frac{2}{1} = 2_{,,}$$

$\therefore$ ΣU_n and ΣV_n behave the Same,,

ΣV_n is convergent

ΣU_n is also Convergent,,

$\dfrac{1 \cdot 2}{3 \cdot 4 \cdot 5} + \dfrac{2 \cdot 3}{4 \cdot 5 \cdot 6} + \dfrac{3 \cdot 4}{5 \cdot 6 \cdot 7} + \ldots\ldots$ Test for convergence

$$\frac{}{3\cdot4\cdot5}+\frac{}{4\cdot5\cdot6}+\frac{}{5\cdot6\cdot7}$$

Convergence for student's practice

Sol n^{th} Term: $\Sigma U_n = \dfrac{n(n+1)}{(n+2)(n+3)(n+4)}$

$\Sigma V_n = \dfrac{1}{n}$ | $\Sigma V_n = \dfrac{1}{n^p}$; p

$p = 1$

$\therefore \Sigma V_n$ is divergent

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\dfrac{n(n+1)}{(n+2)(n+3)(n+4)}}{\dfrac{1}{n}} = 1$$

$\therefore \Sigma U_n$ and ΣV_n behave the Same

Since ΣV_n is divergent, ΣU_n is also divergent //

Test for the convergence.

$$\Sigma\left(\sqrt{n^2+1} - \sqrt{n^2-1}\right)$$

"Rationalize"

Sol $U_n = \sqrt{n^2+1} - \sqrt{n^2-1}$: Rationalize

$$U_n = \sqrt{n^2+1} - \sqrt{n^2-1} \times \frac{\sqrt{n^2+1} + \sqrt{n^2-1}}{\sqrt{n^2+1} + \sqrt{n^2-1}}$$

$(a-b)(a+b) = a^2 - b^2$

$$= \frac{\left(\sqrt{n^2+1}\right)^2 - \left(\sqrt{n^2-1}\right)^2}{\sqrt{n^2+1} + \sqrt{n^2-1}} = \frac{n^2+1 - n^2+1}{\sqrt{n^2+1} + \sqrt{n^2-1}} = \frac{2}{\sqrt{n^2+1} + \sqrt{n^2-1}}$$

$$U_n = \frac{2}{\sqrt{n^2+1} + \sqrt{n^2-1}}$$

$\therefore$ By Comparision Test:

By Auxillary P-Test: $\Sigma V_n = \dfrac{1}{n^p}$

$\Sigma V_n = \dfrac{1}{n}$ | $\Sigma V_n = \dfrac{1}{n^p}$:

$\therefore p = 1$, $\therefore \Sigma V_n$ is divergent

$$\lim \frac{U_n}{} = \lim \frac{2}{\sqrt{n^2+1} + \sqrt{n^2-1}} = \lim \frac{2}{\cancel{n}\left[\sqrt{1+1/n^2} + \sqrt{1-1/n^2}\right]}$$

$$\lim_{n\to\infty}\frac{U_n}{V_n} = \lim_{n\to\infty}\frac{\frac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{2}{\cancel{n}\left[\sqrt{1+1/n^2}+\sqrt{1-1/n^2}\right]}}{\cancel{1/n}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = 1$$

$\therefore$ ΣU_n and ΣV_n behaves the same

Since ΣV_n is divergent.

$\therefore$ ΣU_n is divergent.

Practice

$\sum_{n=1}^{\infty}\left\{\sqrt[3]{n^3+1}-n\right\}$

$\frac{2^q}{1^p}+\frac{3^q}{2^p}+\frac{4^q}{3^p}+\cdots\cdots$

$\Sigma\left[\sqrt{n+1}-\sqrt{n}\right]$

$\sum_{n=1}^{\infty}\left[\sqrt{n^4+1}-\sqrt{n^4-1}\right]$

$\sum_{n=1}^{\infty}\frac{1}{2^n+3^n}$

$\sum\left[\sqrt{n+1}-\sqrt{n}\right]$, Test for the Convergence.

Sol Let us consider, $\sum U_n = \sum\left[\sqrt{n+1}-\sqrt{n}\right] \longrightarrow$ Rationalize,

$$U_n = \sqrt{n+1}-\sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \quad , \quad \because (a-b)(a+b) = a^2-b^2$$

$$U_n = \frac{(\cancel{\sqrt{n+1}})^{\cancel{2}} - (\cancel{\sqrt{n}})^{\cancel{2}}}{\sqrt{n+1}+\sqrt{n}} =$$

$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$U_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$, By comparision Test & Auxillary p-Test

$$\sum V_n = \frac{1}{n^p}$$

$$\sum V_n = \frac{1}{n^{1/2}} \quad \Big| \quad \sum V_n = \frac{1}{n^p}$$

$p = \frac{1}{2}$

$p < 1$, $\sum V_n$ is divergent,

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\frac{1}{\cancel{\sqrt{n}}\left[\sqrt{1+1/n}+1\right]}}{\frac{1}{\cancel{\sqrt{n}}}}$$

$$= \frac{1}{}$$

$$= \frac{1}{\sqrt{1+0}+1} \qquad \left\{ \because \lim_{n\to\infty} \frac{1}{n} = 0 \right\}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

$\therefore \Sigma U_n$ and ΣV_n behave the same.

Since ΣV_n is divergent

then ΣU_n is also divergent //

\# $\frac{1}{4 \cdot 7 \cdot 10} + \frac{4}{7 \cdot 10 \cdot 13} + \frac{9}{10 \cdot 13 \cdot 16} + \cdots$

Test for the Convergence.

Sol

Nr: 1 4 9 ......

$1^2, 2^2, 3^2 \ldots\ldots n^2$ //

$\boxed{1}, 2, 3 \ldots\ldots n^{th} = a + (n-1) \times d = 1 + (n-1) \times 1$

$a = 1$, $d = 1$

$= n$

Nr: $n^{th} = n$

Dr: S-① $\boxed{4}, 7, 10 \ldots\ldots n^{th} = 4 + (n-1) \times 3$

$a = 4$, $d = 3$

$= 4 + 3n - 3$

$= 3n + 1$

S-② $7, 10, 13 \ldots\ldots n^{th}$

$a = 7$, $d = 3$

$= 7 + (n-1) \times 3$

$= 7 + 3n - 3$

$= 3n + 4$

S-③ $10, 13, 16 \ldots\ldots n^{th} = \ldots (n-1) \times 3$

$= 3n+4$

S-3 $10, 13, 16 \ldots\ldots n^{th} =$

a, 3, 3

$10 + (n-1) \times 3$

$= 10 + 3n - 3$

$3n+7$

Dr: $n^{th} = (3n+1)(3n+4)(3n+7)$

$$n^{th} = \frac{Nr}{Dr} = \frac{n^2}{(3n+1)(3n+4)(3n+7)}$$

Let us consider

$$\Sigma U_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)}$$

By Comparision Test & By Aux-p-Test

$$\Sigma V_n = \frac{n^2}{n \times n \times n} = \frac{1}{n}$$

$$\Sigma V_n = \frac{1}{n^p}$$

$$\Sigma V_n = \frac{1}{n} \quad \Big| \quad \Sigma V_n = \frac{1}{n^p}$$

$\therefore p = 1$ $\therefore$ ΣV_n is divergent

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{n^2}{(3n+1)(3n+4)(3n+7)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{n^2}{n \cdot n \cdot n\left[3+\frac{1}{n}\right]\left[3+\frac{4}{n}\right]\left[3+\frac{7}{n}\right]}}{\frac{1}{n}}$$

$$= \frac{1}{(3+0)(3+0)(3+0)} = \frac{1}{27}$$

$\therefore$ ΣU_n and ΣV_n behave the same

Since ΣV_n is divergent, ΣU_n is divergent

D'Alembert's Ratio test

09 April 2025 10:22

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$$

① if $l > 1$, Σu_n is Convergent

② if $l < 1$, Σu_n is divergent

③ if $l = 1$, the Test fails. //

$1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots\dots$ Test for the Convergence.

$2! = 2\times 1$

$3! = 3\times 2\times 1$

$4! = 4\times 3\times 2\times 1$

$n! =$

$(n+1)! = n+1 \times n \times n-1 \times \dots$

$= (n+1)n!$

Sol: $\Sigma u_n = \frac{1}{n!}$, $\Sigma u_{n+1} = \frac{1}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \frac{\frac{1}{n!}}{\frac{1}{(n+1)n!}}$$

$$\frac{u_n}{u_{n+1}} = n+1$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n+1 = \infty$$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$; $l > 1$, Σu_n is Convergent

$l < 1$, " " divergent

$l = 1$, Test fails.

$\Rightarrow \Sigma u_n$ is Convergent //

$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots$ $(x > 0)$

hint Leave the 1st Term,

Sol

$$U_n = \frac{x^n}{n^2+1}, \qquad U_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

$$\frac{U_n}{U_{n+1}} =$$

$$= \frac{\frac{x^n}{n^2+1}}{\frac{x^{n+1}}{(n+1)^2+1}} = \frac{\cancel{x^n}}{n^2+1} \times \frac{(n+1)^2+1}{\cancel{x^n}\cdot x} = \frac{(n+1)^2+1}{n^2+1} \times \frac{1}{x}$$

$$\frac{U_n}{U_{n+1}} = \frac{\left(n\left(1+\frac{1}{n}\right)\right)^2 + 1}{n^2+1} \times \frac{1}{x} = \frac{n^2\left[\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}\right]}{n^2\left[1+\frac{1}{n^2}\right]} \times \frac{1}{x}$$

$$\frac{U_n}{U_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}}{\left(1+\frac{1}{n^2}\right)} \cdot \frac{1}{x}$$

$$\because \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{(1+0)^2+0}{(1+0)} \times \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{1}{1} \times \frac{1}{x} = \frac{1}{x} = l$$

$\therefore$ 1) if $l > 1$, ΣU_n is convergent

$\frac{1}{x} > 1 \Rightarrow x < 1$, ΣU_n is convergent,

2) if $l < 1$, ΣU_n is Divergent

$\frac{1}{x} < 1$, $x > 1$, ΣU_n is divergent

** 3) if $l = 1$, Test fails

3) if $l = 1$, Test fails

$\frac{1}{x} = 1$; $x = 1$, Test fails

Case (ii) if $x = 1$, Test fails, in U_n.

$$\sum U_n = \frac{x^n}{n^2+1} = \frac{1}{n^2+1}$$

By Comparision &
By Auxillary P-Test
$\sum V_n = \frac{1}{n^p}$

$$\sum V_n = \frac{1}{n^2} \quad \Big| \quad \sum V_n = \frac{1}{n^p}$$

$\boxed{p = 2}$

$\therefore$ if $p > 1$, $\sum V_n$ is convergent,

$\therefore$ $\sum U_n$ and $\sum V_n$ behaves the same

Since $\sum V_n$ is convergent, then $\sum U_n$ is also convergent. //

Addition
$p > 1$, $\sum U_n$ converges
$p \leq 1$, diverges

$\boxed{x < 1}$

Test for the convergence of the series of

$$\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots \quad (x > 0).$$

Sol:

Practice for students:

$$U_n = \frac{x^n}{n(n+1)}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{1}{x}$$

✓ $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$

✓ $\sum V_n = \frac{1}{n^2}$

✓ U_n is convergent //

$\sum \frac{1\cdot 2\cdot 3 \cdots\cdots n}{3\cdot 5\cdot 7 \cdots\cdots (2n+1)}$

Sol:

$$\sum U_n = \sum \frac{1\cdot 2\cdot 3 \cdots\cdots n}{3\cdot 5\cdot 7 \cdots (2n+1)}$$

Ratio Test = U_n, U_{n+1}

$n+1^{th}$

$$\sum U_{n+1} = \frac{1\cdot 2\cdot 3 \cdots\cdots n(n+1)}{3\cdot 5\cdot 7 \cdots\cdots (2n+1)(2(n+1)+1)}$$

$n+1^{th}$

$$U_{n+1} = \frac{1\cdot 2\cdot 3 \cdots\cdots n(n+1)}{3\cdot 5\cdot 7 \cdots\cdots (2n+1)(2n+3)}$$

$$\frac{U_n}{U_{n+1}} = \frac{\dfrac{1\cdot 2\cdot 3 \cdots\cdots n}{3\cdot 5\cdot 7 \cdots\cdots 2n+1}}{\dfrac{1\cdot 2\cdot 3 \cdots\cdots n(n+1)}{3\cdot 5\cdot 7 \cdots\cdots (2n+1)(2n+3)}} = \cancel{\frac{1\cdot 2\cdot 3 \cdots n}{3\cdot 5\cdot 7 \cdots (2n+1)}} \times \frac{\cancel{3\cdot 5\cdot 7 \cdots} (2n+1)(2n+3)}{\cancel{1\cdot 2\cdot 3 \cdots n}\,(n+1)}$$

$$\frac{U_n}{U_{n+1}} = \frac{2n+3}{n+1}$$

$\because \lim\limits_{n\to\infty} \frac{1}{n} = 0$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{2n+3}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}[2+3/n]}{\cancel{n}[1+1/n]} = \frac{2+0}{1+0} = 2$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = 2 \;// = l$$

$\therefore\ l > 1$, $\sum U_n$ is convergent //

$\sum^{\infty} \frac{1\cdot 3\cdot 5 \cdots 2n-1}{\cdots} x^{n-1}$

$$\sum_{n=1} \frac{1\cdot3\cdot5 \cdots}{2\cdot4\cdot6\cdots2n} x$$ Practice

$$u_n = \frac{1\cdot3\cdot5 \cdots\cdots 2n-1}{2\cdot4\cdot6 \cdots\cdots 2n} \cdot x^{n-1}$$

$$u_{n+1} = \frac{1\cdot3\cdot5 \cdots 2n-1\,(2n+1)}{2\cdot4\cdot6 \cdots\cdots 2n\;(2n+2)} \cdot x^{n}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} = l$$

1) $l>1$; $x<1$, $\sum u_n$ convergent
2) $l<1$, $x>1$, $\sum u_n$ is divergent
3) $l=1$, $x=1$, Test fails.

$\Rightarrow$ Solve by considering $x=1$
Apply comparisism Test

Test for the Convergence of

$$1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots\cdots \quad (x>0)$$

Sol

$u_n = n^2 x^{n-1}$

$u_{n+1} = (n+1)^2 x^n$

$\frac{u_n}{u_{n+1}} =$

Ratio test cont….

11 April 2025 09:40

\# $\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\dots \infty$

No Extra Scores

Rough: $2, 3, 4 \dots\dots$

$2 + (n-1) \times 1$

$2 + n - 1$

$n+1$

Sol

$$\Sigma U_n = \frac{x^{n+1}}{n+1} \quad \text{——} \; (1)$$

$$\Sigma U_{n+1} = \frac{x^{n+2}}{n+2}$$

$$\frac{U_n}{U_{n+1}} = \frac{x^{n+1}}{n+1} \times \frac{n+2}{x^{n+2}}$$

$$= \frac{\cancel{x^n} \cdot \cancel{x}}{\cancel{x^n} \cdot \cancel{x^2}} \times \frac{n+2}{n+1}$$

$$\frac{U_n}{U_{n+1}} = \frac{1}{x} \times \frac{n+2}{n+1}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{n+2}{n+1} \times \frac{1}{x} = \lim_{n\to\infty} \frac{n\,[1+2/n]}{n\,[1+1/n]} \times \frac{1}{x}$$

$\because \lim_{n\to\infty} \frac{1}{n} = 0$

$$= \frac{1}{1} \times \frac{1}{x} = \frac{1}{x} = l$$

By D'Alembert's Ratio Test,

1) if $l > 1$, ΣU_n is Convergent

$\frac{1}{x} > 1$, $x < 1$, ΣU_n is convergent,

2) if $l < 1$, ΣU_n is divergent

$\frac{1}{x} < 1$, $x > 1$, series is divergent

3) if $l = 1$, Test fails

3) if $l = 1$, Test fails

$\frac{1}{x} = 1$, $\boxed{x = 1}$ Test fails.

when $x = 1$, in eq...(1)

$$U_n = \frac{x^{n+1}}{n+1}\Bigg|_{x=1} = \frac{1}{n+1}$$

By Comparision Test &

By Auxillary p-Test

$\Sigma V_n = \frac{1}{n^p}$

$$\Sigma U_n = \frac{1}{n\left[1 + 1/n\right]}$$

$$\Sigma V_n = \frac{1}{n} \qquad \Big| \qquad \Sigma V_n = \frac{1}{n^p}$$

$\therefore \boxed{p = 1}$, ΣV_n is divergent

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n(1+1/n)}}{\frac{1}{n}} = 1$$

$\therefore$ ΣU_n and ΣV_n behaves the same

Since ΣV_n is divergent, then ΣU_n is also divergent

\# Test the Convergence of Series.

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdots\cdots (2n+1)}{2\cdot 5\cdot 8 \cdots\cdots (3n+2)}$$

Sol:

Cauchy's nth root test

11 April 2025 09:59

if $\sum U_n$ is a series of +ve terms such that

$$\lim_{n\to\infty} (U_n)^{1/n} = l \text{, then}$$

a) $\sum U_n$ Converges, if $l < 1$

b) $\sum U_n$ diverges, if $l > 1$

c) Test fails to decide the nature, if $l = 1$

(Here $(U_n)^{1/n}$ stands for the positive n^{th} root of U_n)

#1. Test for the Convergence of $\sum \left[1+\frac{1}{n}\right]^{-n^2}$

Sol Let us consider.

$$\sum U_n = \sum \left[1+\frac{1}{n}\right]^{-n^2}$$

$$U_n = \left[1+\frac{1}{n}\right]^{-n^2} \; ; \; (U_n)^{1/n} = \left[\left[1+\frac{1}{n}\right]^{-n^2}\right]^{1/n}$$

$$= (U_n)^{1/n} = \left[1+\frac{1}{n}\right]^{-n}$$

$$(U_n)^{1/n} = \frac{1}{\left[1+\frac{1}{n}\right]^n}$$

$\because \lim_{n\to\infty} \left[1+\frac{1}{n}\right]^n = e$

$\because e = 2.71$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left[\frac{1}{\left[1+\frac{1}{n}\right]^n}\right] = \frac{1}{e}$$

$$= \frac{1}{2.71} = 0.36 = l$$

$\therefore l < 1$, $\sum U_n$ is Convergent //

$\therefore\ l < 1,\ \sum u_n$ is Convergent //

$\therefore$ By Cauchy's n^{th} Root, $\sum u_n$ Converges //

#2. Test for the convergence of $\sum \dfrac{1}{(\log n)^n}$

Sol hint: $\log \infty = \infty$ (Practice for students)

#3 Test for the Convergence of $\sum \left(\dfrac{n+1}{n+2}\right)^n \cdot x^n \quad (x>0)$

Sol Let us Consider

$$\sum u_n = \left(\frac{n+1}{n+2}\right)^n \cdot x^n$$

$$(u_n)^{1/n} = \left[\left[\left(\frac{n+1}{n+2}\right)\cdot x\right]^n\right]^{1/n} = \frac{n+1}{n+2} \times x$$

$$(u_n)^{1/n} = \frac{n\left[1+1/n\right]}{n\left[1+2/n\right]}\cdot x$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{(1+1/n)}{(1+2/n)}\cdot x = x = l$$

$\sqrt{625}$ = $\boxed{25}$

By Cauchy's n^{th} root Test.

1) $l < 1$, $\sum u_n$ Convergent

$\Rightarrow x < 1$, $\sum u_n$ Convergent

2) $l > 1$, $\sum u_n$ divergent

$x > 1$, $\sum u_n$ divergent

3) $l = 1$, Test fails

$x = 1$, Test fails.

When

when

$x = 1$ in U_n

$$U_n = \left(\frac{n+1}{n+2}\right)^n \cdot x^n \Bigg|_{x=1} = \left(\frac{n+1}{n+2}\right)^n$$

$$= \frac{n^n(1+1/n)^n}{n^n(1+2/n)^n}$$

$$\lim_{n\to\infty} U_n = \frac{\lim_{n\to\infty}(1+1/n)^n}{\lim_{n\to\infty}(1+2/n)^n} = \frac{e}{e^2} = \frac{1}{e}$$

$$= \underline{\underline{0.36}} < 1$$

$\therefore U_n$ is divergent //

**** $\lim_{n\to\infty} U_n =$

$= l > 1$, Cnvrgnt

$l \leq 1$, dvrgnt

$\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2$

$\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3$

$e = 2.71$

Test for convergenes of $\sum \frac{1}{n^n}$

Quick review + RAABE'S TEST

16 April 2025 09:37

Name of Test	Condition's
✓ Limit Comparision Test	
$\sum V_n = \frac{1}{n^p}$ (Auxillary P-test	$p>1$, $\sum V_n$ is convergent $p \le 1$, $\sum V_n$ is divergent

$$\lim_{n\to\infty} \frac{u_n}{v_n} = (l)$$

$\sum U_n$ and $\sum V_n$ behaves the same

$\sum V_n$, U_n follows

✓ D'Alembert's ratio Test

$U_n = ?$

U_{n+1}

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$$

Conditions

1) $l > 1$, $\sum U_n$ is Convergent
2) $l < 1$, $\sum U_n$ is divergent
3) $l = 1$, Test fails

✓ Cauchy's n^{th} root Test

$U_n = ?$

$$\lim_{n\to\infty} (U_n)^{1/n} = l$$

Condition's (Root = X Ratio) opp

1) $l > 1$, $\sum U_n$ is divergent
2) $l < 1$, Convergent
3) $l = 1$, Test fails

RAABE'S TEST

$$\boxed{\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = l}$$

S/L $\boxed{R_R}$ //

... the series is convergent

$$n\to\infty \quad \left[u_{n+1} \right] \qquad [K]$$

1) if $l > 1$, the series is convergent

2) if $l < 1$, the series is divergent

3) if $l = 1$, Test fails.

Test for Convergence of the series

$$x^2 + \frac{2^2}{3\cdot 4}x^4 + \frac{2^2 4^2}{3\cdot 4\cdot 5\cdot 6}x^6 + \frac{2^2 4^2 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8 + \cdots\cdots$$

Sol:

Here: $\Sigma U_n = \frac{(2n)^2}{(2n+1)(2n+2)} \cdot x^{2n+2}$

$$\Sigma U_n = \frac{2^2 4^2 6^2 \cdots\cdots (2n)^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdots\cdot (2n+1)(2n+2)} \cdot x^{2n+2}$$

$$U_{n+1} = \frac{2^2 4^2 \cdot 6^2 \cdots\cdots (2n)^2 (2n+2)^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdots (2n+1)(2n+2)(2n+3)(2n+4)} \cdot x^{2n+4}$$

$$\frac{U_n}{U_{n+1}} = \frac{2^2 4^2 6^2 \cdots\cdots (2n)^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdots (2n+1)(2n+2)} \cdot x^{2n}\cdot x^2 \times \frac{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdots (2n+1)(2n+2)(2n+3)(2n+4)}{2^2 4^2 6^2 \cdots (2n)^2 (2n+2)^2} \times \frac{1}{x^{2n}\cdot x^4}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)(2n+4)}{(2n+2)^2} \times \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(2n+3)(2n+4)}{(2n+2)^2} \times \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2} = l$$

$\therefore$ By D'Alembert's ratio Test,

(i) If $l > 1$, Σu_n is convergent $\quad x \approx -1, \; -1/2, \; -2$

(i) If $l > 1$, Σu_n is convergent

$\frac{1}{x^2} > 1$, $x^2 < 1$, $x < 1$, Σu_n is convergent

(ii) if $l < 1$, Σu_n is divergent

$x = 1, 2, \dots$ $-1/2, -1/3, \dots$

$\frac{1}{x^2} < 1$, $x^2 > 1$, $x > 1$, Σu_n is divergent

(iii) if $l = 1$, Test fails

$\frac{1}{x^2} = 1$, $x^2 = 1$, $x = 1$, Test fails,,

when $x^2 = 1$ in $\left[\frac{u_n}{u_{n+1}}\right]$

Rabbit

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)(2n+4)}{(2n+2)^2} \times \frac{1}{x^2} \Bigg|_{x^2=1}$$

$$\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)(2n+4)}{(2n+2)^2}$$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{\cancel{4n^2} + \cancel{8n} + 6n + 12 - (\cancel{4n^2} + 4 + \cancel{8n})}{(4n^2 + 4 + 8n)}$$

$$\lim_{n\to\infty}\left[\quad\right]$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{6n+8}{4n^2+4+8n}$$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{6n+8}{4n^2+4+8n}\right] = \left[\frac{6n^2+8n}{4n^2+4+8n}\right]$$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty}\left[\frac{6n^2+8n}{4n^2+4+8n}\right] = \lim_{n\to\infty}\left[\frac{6+8/n}{4+4/n^2+8/n}\right]$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\left[\frac{\text{[illegible]}}{4n^2+4+8n}\right] = \lim_{n\to\infty}\left[\frac{\text{[illegible]}}{4+4/n^2+8/n}\right]$$

$$= \frac{6+0}{4+0+0}$$

$$= \frac{3}{2} > 1$$

$$\left\{\because \lim_{n\to\infty}\frac{1}{n}=0,\quad \lim_{n\to\infty}\frac{1}{n^2}=0\right\}$$

$\therefore$ By Raabe's Test, Series is Convergent //

\# Feb/mar | 2024 | LAQ | O.U | 6 marks

\# Test for the Convergence of $\sum_{n=1}^{\infty} \frac{4\cdot 7\cdot 10 \cdots (3n+1)}{1\cdot 2\cdot 3 \cdots n} x^n \ (x>0)$

INTEGRAL TEST

16 April 2025 10:43

$$\sum_{n=1}^{\infty} f(n) \longrightarrow \lim_{t\to\infty} f(t)$$

$\hookrightarrow f(x)$

$\hookrightarrow f(t) = \int_1^t f(x)\,dx$

\# Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges. ; $n \in [1, \infty)$

Sol Let $f(x) = \frac{1}{x^2+1}$ for $x \in [1, \infty)$

then

$$f(t) = \int_1^t f(x)\,dx$$

$$= \int_1^t \frac{1}{x^2+1}\,dx$$

$\because \int \frac{1}{x^2+1}\,dx = \tan^{-1} x$

$\because \tan^{-1}(1) = \frac{\pi}{4}$

$$f(t) = \left.\operatorname{Tan}^{-1} x\right]_1^t = \operatorname{Tan}^{-1}(t) - \operatorname{Tan}^{-1}(1)$$

$$= \operatorname{Tan}^{-1}(t) - \frac{\pi}{4}$$

$f(t)$

$$\therefore f(t) = \operatorname{Tan}^{-1}(t) - \frac{\pi}{4}$$

$\because \tan^{-1}(+\infty) = \frac{\pi}{2}$

Now $$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[\operatorname{Tan}^{-1}(t) - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \,//$$

$\therefore$ By Integral Test, converges to $\frac{\pi}{4}$ //

$\therefore \sum f(n)$ Converges By Integral test //

ALTERNATING SERIES

16 April 2025 10:06

An Alternating Series may be written as

$$u_1 - u_2 + u_3 - u_4 \cdots\cdots + (-1)^{n-1} u_n$$

where u_n is positive

it is denoted by $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$

Examples

#1, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \times \frac{1}{n}$

#2, $1 - \frac{2}{\log 2} + \frac{3}{\log 3} - \frac{4}{\log 4} + \cdots\cdots = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \times \frac{n}{\log n}$

LEIBNITZ'S TEST

if $\{u_n\}$ is a sequence of +ve Terms such that

(a). $u_1 \geq u_2 \geq u_3 \geq \cdots\cdots \geq u_n \geq u_{n+1}$

(b). $\lim_{n\to\infty} u_n = 0$

then, the Alternating series is $\sum_{n=1}^{\infty} (-1)^{n-1} \times u_n$ is convergent.

P.T $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\cdots$ converges.

Sol This is in alternating Series and $u_n = \frac{1}{n!}$ 2m

(i) clearly $u_n > 0$ and $u_n > u_{n+1}$ $\forall$ 'n'

also

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n!} = 0$

$\therefore$ By Leibnitz Test , $\sum \frac{(-1)^n}{n!}$ converges //

\# Examine the Convergence of $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \cdots$

Practice

\# Feb/mar/2024/O-U/SAQ/2m

Discuss the Convergence of the series $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots$

Absolute and conditional convergence

16 April 2025 10:21

5.29(a). ABSOLUTE CONVERGENCE OF A SERIES (P.T.U., Dec. 2003, 2011)

Def. *If a convergent series whose terms are not all positive, remains convergent when all its terms are made positive, then it is called an* **absolutely convergent series**, *i.e.,*

The series Σu_n is said to be absolutely convergent if $\Sigma |u_n|$ is a convergent series.

5.29(b). CONDITIONAL CONVERGENCE OF A SERIES (P.T.U., Dec. 2011)

A series is said to be ***conditionally convergent*** *if it is convergent but does not converge absolutely.*

Example 1. *Test whether the following series are absolutely convergent or conditionally convergent.*

(a) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ (P.T.U., Dec. 2006) (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$. (P.T.U., Dec. 2012)

a) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ check for absolute / Conditional convergence

Sol This is alternating series, with $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n^2}$

in order to Apply Leibnitz Test,

(i) $u_n > u_{n+1}$ is satisfied from the series.

(ii) Apply $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$ $\left\{ \because \lim_{n\to\infty} \frac{1}{n^2} = 0 \right\}$

$\therefore$ By Leibnitz's Test, Σu_n of Series is $\boxed{\text{Convergent}}$

Now

$\Sigma |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$

Now

$|u_n| = \frac{1}{n^2}$ By Comparision Test, $\Sigma v_n = \frac{1}{n^p}$

$\Sigma v_n = \frac{1}{n^2}$; $\Sigma v_n = \frac{1}{n^p}$; $\boxed{p = 2}$, $p > 1$

$\Sigma |u_n|$ series is $\boxed{\text{Convergent}}$

Since, Σu_n and $\Sigma |u_n|$ both are convergent, $\therefore$

$\therefore$ Series is ABSOLUTE Convergence //